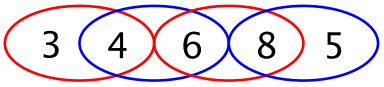

Problem of the Week
Problem E and Solution
Linked Digits

Problem

The number 34 685 contains 5 digits and we say its digit length is 5. The left-most digit of 34 685 is 3. Each pair of consecutive digits, 34, 46, 68 and 85, forms a number which is divisible by either 17 or 23. The left-most digit of an integer with digit length 100 is 3. This integer also has the property that any pair of consecutive digits forms a number which is divisible by either 17 or 23. How many such integers are there?

Solution

To do this problem we need to know the two-digit multiples of 17 and 23.
The two-digit multiples of 17 are 17, 34, 51, 68 and 85.
The two-digit multiples of 23 are 23, 46, 69 and 92.

Since the integer starts with 3 and the only two-digit number in the two lists starting with 3 is 34, the second digit is 4. Similarly, the third digit is 6 since the only two-digit number in the two lists starting with 4 is 46. The fourth digit, however, can be 8 or 9 since there are two two-digit numbers, 68 and 69, from the two lists that start with 6. Now we need to consider two cases.

Case 1: The fourth digit is 8.

Since the only two-digit number in the two lists starting with 8 is 85, the fifth digit must then be 5. Similarly, the sixth digit is 1 and the seventh is 7. We must stop here. There is no two-digit number in either list that starts with a 7. Therefore, the fourth digit must not be 8.

Case 2: The fourth digit is 9.

Since the only two-digit number in the two lists starting with 9 is 92, the fifth digit is 2. Similarly, the sixth digit is 3. We can now repeat our argument from the beginning. The digit after the 3 must be a 4, then a 6, then a 9, then a 2. The five digits '34692' will continue to repeat as long as 9 follows 6. So the number will be 34692 34692 34692 34 . . ..

Since $100 \div 5 = 20$, there will be 20 blocks of 5 digits. The first 19 must be 34692. The last 5 digits, however, can be 34692 or 34685.

Therefore, there are two integers with left-most digit 3 and digit length 100 such that any pair of consecutive digits form a number which is divisible by 17 or 23.

For completeness, the two numbers are:

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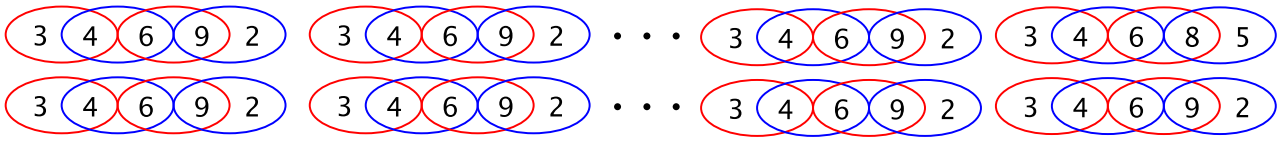
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