



Problem of the Week

Problem D and Solution

PRODUCTivity

Problem

The “*digit product*” of a positive integer is the product of the individual digits of the integer. For example, the digit product of 234 is $2 \times 3 \times 4 = 24$. Other numbers also have a digit product of 24. For example, 2223, 113 181 and 38 each have a digit product of 24. The number 38 is the smallest positive integer with a digit product of 24. There are many positive integers whose digit product is 2000. Determine the smallest positive integer whose digit product is 2000.

Solution

Let N be the smallest positive integer whose digit product is 2000.

In order to find N , we must find the minimum possible number of digits whose product is 2000. This is because if the integer a has more digits than the integer b , then $a > b$.

Once we have determined the digits that form N , then the integer N is formed by writing those digits in increasing order.

Note that the digits of N cannot include 0, or else the digit product of N would be 0.

Also, the digits of N cannot include 1, otherwise we could remove the 1 and obtain an integer with fewer digits (and thus, a smaller integer) with the same digit product. Therefore, the digits of N will be between 2 and 9, inclusive.

Since the digit product of N is 2000, we will use the prime factorization of 2000 to help determine the digits of N :

$$2000 = 2^4 \times 5^3$$

In order for a digit to have a factor of 5, the digit must equal 5. Therefore, three of the digits of N are 5.

The remaining digits of N must have a product of $2^4 = 16$. We need to find a combination of the smallest number of digits whose product is 16. We cannot have one digit whose product is 16, but we can have two digits whose product is 16. In particular, $16 = 2 \times 8$ and $16 = 4 \times 4$.

Therefore, N has 5 digits. They are 5, 5, 5, 2, 8 or 5, 5, 5, 4, 4. In order for N to be as small as possible, its digits must be in increasing order. The smallest possible number formed by the digits 5, 5, 5, 2, 8 is 25 558. The smallest possible number formed by the digits 5, 5, 5, 4, 4 is 44 555.

Since $25\ 558 < 44\ 555$, the smallest N is 25 558. That is, the smallest positive integer with a digit product of 2000 is 25 558.

