The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2016 Canadian Senior Mathematics Contest

Wednesday, November 23, 2016
(in North America and South America)

Thursday, November 24, 2016
(outside of North America and South America)

Solutions

Part A

1. Evaluating,

$$\frac{1^3+2^3+3^3}{(1+2+3)^2} = \frac{1+8+27}{6^2} = \frac{36}{36} = 1$$

Answer: 1

2. *Solution 1*

Suppose that the full price of the second book was $\$x$.
Since 50% is equivalent to $\frac{1}{2}$, then the amount that Mike paid, in dollars, for the two books was $33 + \frac{1}{2}x$.
The combined full price, in dollars, of the two books was $33 + x$.
Since Mike saved a total of 20% on his purchase, then he paid 80% of the combined full cost.
Since 80% is equivalent to $\frac{4}{5}$, then, in dollars, Mike paid $\frac{4}{5}(33 + x)$.
Therefore, we obtain the equivalent equations

$$\begin{aligned} \tfrac{4}{5}(33+x) &= 33 + \tfrac{1}{2}x \\ \tfrac{4}{5}(33) + \tfrac{4}{5}x &= 33 + \tfrac{1}{2}x \\ \tfrac{3}{10}x &= \tfrac{1}{5}(33) \\ x &= \tfrac{10}{3} \cdot \tfrac{1}{5}(33) \\ x &= 2 \cdot 11 = 22 \end{aligned}$$

Therefore, the full cost was $\$(33 + x) = \$(33 + 22) = \$55$.
Mike saved 20% of the full cost or $\frac{1}{5}(\$55) = \$11$.

Solution 2

Suppose that the full price of the second book was $\$x$.
The combined full price, in dollars, of the two books was $33 + x$.
Since 50% is equivalent to $\frac{1}{2}$, then the amount that Mike saved, in dollars, on the second book was $\frac{1}{2}x$.
Since 20% is equivalent to $\frac{1}{5}$, then, in dollars, Mike saved a total of $\frac{1}{5}(33 + x)$ on the purchase of both books.
Therefore, we obtain the equivalent equations

$$\begin{aligned} \tfrac{1}{5}(33+x) &= \tfrac{1}{2}x \\ 2(33+x) &= 5x \\ 66 + 2x &= 5x \\ 3x &= 66 \\ x &= 22 \end{aligned}$$

Therefore, the full cost of the second book was \$22.
Mike saved 50% of the full cost or $\frac{1}{2}(\$22) = \$11$.

Answer: \$11

3. There are exactly 5 lists with this property, namely

$$3+5+7+9=1+5+7+11=1+3+5+15=1+3+7+13=1+3+9+11=24$$

Why are these the only lists?

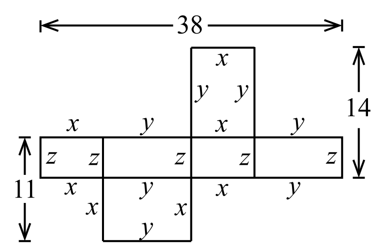
If $a=3$, then since a,b,c,d are odd positive integers with $a<b<c<d$, then $b\geq 5$ and $c\geq 7$ and $d\geq 9$, and so $a+b+c+d\geq 3+5+7+9=24$.
Since $a+b+c+d=24$, then $b=5$ and $c=7$ and $d=9$.
If $a=1$, then we cannot have $b\geq 7$, since otherwise $c\geq 9$ and $d\geq 11$, which gives

$$a+b+c+d\geq 1+7+9+11=28$$

Therefore, if $a=1$, we must have $b=3$ or $b=5$ since $a<b$.
If $b=5$, then $c+d=24-a-b=24-1-5=18$.
Since we have $5<c<d$, then we must have $18=c+d>2c$ and so $c<9$.
Since c and d are odd positive integers, then $c=7$ and so $d=11$.
If $b=3$, then $c+d=24-1-3=20$.
Therefore, using a similar argument, we get $c<10$ and so $c=9$ (which gives $d=11$) or $c=7$ (which gives $d=13$) or $c=5$ (which gives $d=15$).
Therefore, there are exactly 5 lists with the desired properties.

Answer: 5

4. Suppose that the rectangular prism has dimensions $x\times y\times z$.
We label the various lengths as shown:



(We can think about the four "middle" rectangles in the net as forming the sides of the prism, the "top" rectangle as forming the top, and the "bottom" rectangle as forming the bottom.)
From the given information, we obtain the equations $2x+2y=38$ and $y+z=14$ and $x+z=11$.
Dividing the first equation by 2, we obtain $x+y=19$.
Adding the three equations, we obtain $(x+y)+(y+z)+(x+z)=19+14+11$.
This gives $2x+2y+2z=44$ or $x+y+z=22$.
Therefore,

$$\begin{aligned} x&=(x+y+z)-(y+z)=22-14=8\\ y&=(x+y+z)-(x+z)=22-11=11\\ z&=(x+y+z)-(x+y)=22-19=3 \end{aligned}$$

Finally, the volume of the prism is thus $xyz=8\cdot 11\cdot 3=264$.

Answer: 264

5. Since the first player to win 4 games becomes the champion, then Gary and Deep play at most 7 games. (The maximum number of games comes when the two players have each won 3 games and then one player becomes the champion on the next (7th) game.)
We are told that Gary wins the first two games.
For Deep to become the champion, the two players must thus play 6 or 7 games, because Deep wins 4 games and loses at least 2 games. We note that Deep cannot lose 4 games, otherwise Gary would become the champion.
If Deep wins and the two players play a total of 6 games, then the sequence of wins must be GGDDDD. (Here D stands for a win by Deep and G stands for a win by Gary.)
If Deep wins and the two players play a total of 7 games, then Deep wins 4 of the last 5 matches and must win the last (7th) match since he is the champion.
Therefore, the sequence of wins must be GGGDDDD or GGDGDDD or GGDDGDD or GGDDDGD. (In other words, Gary can win the 3rd, 4th, 5th, or 6th game.)
The probability of the sequence GGDDDD occurring after Gary has won the first 2 games is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. This is because the probability of a specific outcome in any specific game is $\frac{1}{2}$, because each player is equally likely to win each game, and there are 4 games with undetermined outcome.
Similarly, the probability of each of the sequences GGGDDDD, GGDGDDD, GGDDGDD, and GGDDDGD occurring is $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.
Therefore, the probability that Gary wins the first two games and then Deep becomes the champion is $\frac{1}{16} + 4 \cdot \frac{1}{32} = \frac{6}{32} = \frac{3}{16}$.

Answer: $\frac{3}{16}$

6. We label the digits of n in order from left to right as $n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9$. Since n is a nine-digit integer, then $n_1 \neq 0$.
From the given information, the following sums must be multiples of 5: $n_1 + n_2 + n_3 + n_4 + n_5$ and $n_2 + n_3 + n_4 + n_5 + n_6$ and $n_3 + n_4 + n_5 + n_6 + n_7$ and $n_4 + n_5 + n_6 + n_7 + n_8$ and $n_5 + n_6 + n_7 + n_8 + n_9$.
Also, the following sums must be multiples of 4: $n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7$ and $n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8$ and $n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9$.

We need the following fact, which we label $(*)$:

> If a and b are positive integers that are both multiples of the positive integer d, then their difference $a - b$ is also a multiple of d.

This is true because if $a = ed$ and $b = fd$ for some positive integers e and f, then $a - b = ed - fd = d(e - f)$ which is a multiple of d.

By $(*)$, $(n_1 + n_2 + n_3 + n_4 + n_5) - (n_2 + n_3 + n_4 + n_5 + n_6) = n_1 - n_6$ must be a multiple of 5.
Since n_1 and n_6 are taken from the list $0, 1, 2, 3, 4, 5, 6, 7, 8$ and no pair of numbers from this list differs by more than 8, then for $n_1 - n_6$ to be a multiple of 5, we must have that n_1 and n_6 differ by 5.
Similarly, n_2 and n_7 differ by 5, n_3 and n_8 differ by 5, and n_4 and n_9 differ by 5.
The pairs from the list $0, 1, 2, 3, 4, 5, 6, 7, 8$ that differ by 5 are $0, 5$ and $1, 6$ and $2, 7$ and $3, 8$. These integers must be assigned in some order to the pairs n_1, n_6 and n_2, n_7 and n_3, n_8 and n_4, n_9.
Note that n_5 is not in the list of positions affected and 4 is not in the list of integers that can be assigned to these slots.
Therefore, $n_5 = 4$.

Using $(*)$ again, we also see that n_1 and n_8 differ by a multiple of 4, and n_2 and n_9 differ by a multiple of 4.

Suppose that $n_1 = 1$. Then from above, $n_6 = 6$ (using divisibility by 5) and $n_8 = 5$ (using divisibility by 4, since the only integer from our list that differs from 1 by a multiple of 4 is 5).
Since $n_8 = 5$, then using divisibility by 5, we have $n_3 = 0$.
So if $n_1 = 1$, the digits of n look like 1_0_46_5_.
Using similar analysis, we make a table that lists the first digit, and the resulting configurations:

n_1	Configuration
1	1_0_46_5_
2	2_1_47_6_
3	3_2_48_7_
5	5_6_40_1_
6	6_7_41_2_
7	7_8_42_3_
8	8_5_43_0_

Note that, in the case where $n_1 = 8$, we know that n_8 differs from 8 by a multiple of 4, but cannot equal 4 since $n_5 = 4$ already.

While it is necessary that n_1 and n_8 differ by a multiple of 4, it is not sufficient to guarantee that the sum of each set of 7 consecutive digits is a multiple of 4.
Since the sum of the 9 digits of n is $0+1+2+3+4+5+6+7+8 = 36$ (which is a multiple of 4) and $n_3+n_4+n_5+n_6+n_7+n_8+n_9$ is a multiple of 4, then n_1+n_2 must be a multiple of 4.
Since each of the digits is between 0 and 8, then the maximum sum of two digits is 15, and so the possible multiples of 4 to which $n_1 + n_2$ can be equal are 4, 8 and 12.
If $n_1 = 1$, then this means that $n_2 = 3$ or $n_2 = 7$.
If $n_1 = 2$, then $n_2 = 6$, which is not possible, since $n_8 = 6$.
If $n_1 = 3$, then $n_2 = 1$ or $n_2 = 5$.
If $n_1 = 5$, then $n_2 = 3$.
If $n_1 = 6$, then $n_2 = 2$, which is not possible, since $n_8 = 6$.
If $n_1 = 7$, then $n_2 = 1$ or $n_2 = 5$.
If $n_1 = 8$, then $n_2 = 0$ or $n_2 = 4$, which are not possible, as these digits are already used.

If $n_2 = 3$, then using a similar analysis to that above, we have $n_7 = 8$ and $n_9 = 7$ and $n_4 = 2$.
Thus, when $n_1 = 1$ and $n_2 = 3$, then $n = 130246857$.
Using similar analysis on the remaining cases, we obtain the following possible values for n:

$$130246857, 170846253, 312048675, 352648071, 536240817, 576840213, 718042635, 758642031$$

We can check that each of these nine-digit integers has the desired properties.
The sum of these eight nine-digit integers is 3555555552.
(We note that the sum of the possible values for each digit is 32.)

ANSWER: 3555555552

Part B

1. (a) When the table is continued, we obtain

Palindrome	Difference
1001	
	110
1111	
	110
1221	
	110
1331	
	110
1441	
	110
1551	
	110
1661	
	110
1771	
	110
1881	
	110
1991	
	11
2002	
	110
2112	
	⋮
⋮	

The eighth and ninth palindromes in the first column are 1771 and 1881.

(b) When we continue to calculate positive differences between the palindromes in the first column, we see that

$$1551 - 1441 = 1661 - 1551 = 1771 - 1661 = 1881 - 1771 = 1991 - 1881 = 110$$

but that $2002 - 1991 = 11$.
Since we are told that there are only two possible differences, then $x = 11$.

(c) *Solution 1*
The palindromes between 1000 and 10 000 are exactly the positive integers of the form $xyyx$ where x and y are digits with $1 \leq x \leq 9$ and $0 \leq y \leq 9$. (We note that each integer is a palindrome and that every palindrome in the desired range has four digits and so is of this form.)
Since there are 9 choices for x and 10 choices for y, then there are $9 \times 10 = 90$ palindromes in the first column.
That is, $N = 90$.

Solution 2
The palindromes between 1000 and 10 000 are exactly all of the four-digit palindromes.
Between 1000 and 2000, there are 10 palindromes, as seen in the table above.
Between 2000 and 3000, there are also 10 palindromes:

$$2002, 2112, 2222, 2332, 2442, 2552, 2662, 2772, 2882, 2992$$

Similarly, between 3000 and 4000, there are 10 palindromes, as there are between 4000 and 5000, between 5000 and 6000, between 6000 and 7000, between 7000 and 8000, between 8000 and 9000, and between 9000 and 10 000.
In other words, in each of these 9 ranges, there are 10 palindromes, and so there are $9 \times 10 = 90$ palindromes in total.
That is, $N = 90$.

(d) Since there are 90 palindromes in the first column, there are 89 differences in the second column.
The difference between two consecutive palindromes with the same thousands digit is 110. This is because two such palindromes can be written as $xyyx$ and $xzzx$, where $z = y + 1$. When these numbers are subtracted, we obtain "0110" or 110.
Since there are 9 groups of 10 palindromes with the same thousands digit, then there are $10 - 1 = 9$ differences equal to 110 in each of these 9 groups. This represents a total of $9 \times 9 = 81$ differences.
Since there are 89 differences and each is equal to 110 or to 11 (from (b)), then there are $89 - 81 = 8$ differences equal to 11.
(We can check that the difference between the consecutive palindromes $x99x$ and $w00w$ (where $w = x + 1$) is always 11.)
Thus, the average of the 89 numbers in the second column is

$$\frac{81 \cdot 110 + 8 \cdot 11}{89} = \frac{8910 + 88}{89} = \frac{8998}{89}$$

2. (a) To find the coordinates of P, which is on the x-axis, we set $y = 0$ to obtain the equivalent equations

$$\begin{aligned} (x-6)^2 + (0-8)^2 &= 100 \\ (x-6)^2 + 64 &= 100 \\ (x-6)^2 &= 36 \\ x - 6 &= \pm 6 \end{aligned}$$

Therefore, $x - 6 = -6$ (or $x = 0$) or $x - 6 = 6$ (or $x = 12$).
Since O has coordinates $(0, 0)$, then P has coordinates $(12, 0)$.

(b) *Solution 1*
The point, Q, with the maximum y-coordinate is vertically above the centre of the circle, C.
Since C has coordinates $(6, 8)$ and the radius of the circle is 10, then Q has coordinates $(6, 8 + 10)$ or $(6, 18)$.

Solution 2
Consider a point (x, y) on the circle.
Since $(x-6)^2 + (y-8)^2 = 100$, then $(y-8)^2 \leq 100 - (x-6)^2$.
Since $y > 8$, we can maximize y by maximizing $(y-8)^2$.
Since $(x-6)^2 \geq 0$, then the maximum possible value of $(y-8)^2$ is 100, when $(x-6)^2 = 0$ or $x = 6$.
Since $y > 8$, then $y - 8 = 10$ or $y = 18$.
This means that the point with the maximum y-coordinate is $Q(6, 18)$.

(c) *Solution 1*
Since $\angle PQR = 90^\circ$, then a property of circles tells us that PR is a diameter of the circle.
Also, if PR is a diameter of the circle, then C (the centre of the circle) is its midpoint.
Since P has coordinates $(12, 0)$ and C has coordinates $(6, 8)$, then if the coordinates of R are (a, b), we must have $6 = \frac{1}{2}(12 + a)$ (which gives $a = 0$) and $8 = \frac{1}{2}(b + 0)$ (which gives $b = 16$).
Therefore, the coordinates of R are $(0, 16)$.

Solution 2

Since P has coordinates $(12, 0)$ and Q has coordinates $(6, 18)$, then the slope of PQ is $\frac{18-0}{6-12} = -3$.

Since $\angle PQR = 90^\circ$, then PQ and QR are perpendicular.

Since the slope of PQ is -3, then the slope of QR is the negative reciprocal of -3, which is $\frac{1}{3}$.

Since Q has coordinates $(6, 18)$, then the equation of the line through Q and R is $y - 18 = \frac{1}{3}(x - 6)$ or $y = \frac{1}{3}x + 16$.

We want to determine the second point of intersection of the line with equation $y = \frac{1}{3}x+16$ with the circle with equation $(x - 6)^2 + (y - 8)^2 = 100$.

Substituting for y, we obtain

$$\begin{aligned}
(x-6)^2 + \left(\tfrac{1}{3}x + 16 - 8\right)^2 &= 100 \\
9(x-6)^2 + 9\left(\tfrac{1}{3}x + 8\right)^2 &= 900 \\
9(x^2 - 12x + 36) + (x + 24)^2 &= 900 \\
9x^2 - 108x + 324 + x^2 + 48x + 576 &= 900 \\
10x^2 - 60x &= 0 \\
10x(x-6) &= 0
\end{aligned}$$

Therefore, $x = 6$ (which gives the point Q) or $x = 0$.

When $x = 0$, we obtain $y = 16$, and so the coordinates of the point R are $(0, 16)$.

(d) *Solution 1*

Suppose that X is a point on the circle for which $\angle PQX = 45^\circ$.

PX is thus a chord of the circle that subtends an angle of 45° on the circumference.

Therefore, $\angle PCX = 2(45^\circ) = 90^\circ$. That is, the central angle of chord PX (which equals twice the subtended angle) is 90° and so XC is perpendicular to PC.

We now show two different ways to find the coordinates of the possible locations for X.

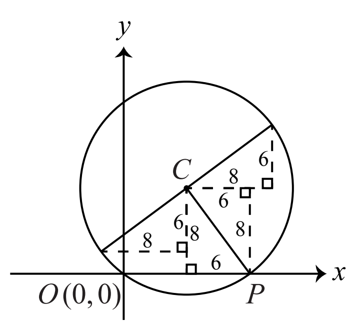
Method 1

Since C has coordinates $(6, 8)$ and P has coordinates $(12, 0)$, then to get from C to P, we move right 6 units and down 8 units.

Since X is on the circle, then $CX = CP$, because both are radii.

By using the same component distances, the length CX will equal the length of CP. By interchanging the component distances and reversing the direction of one of these distances, we obtain a perpendicular slope. (In the diagram below, we can how how copies of the right-angled triangle with side lengths 6, 8 and 10 fit together to form these right angles.)

Since CX is perpendicular to CP, then possible locations for point X can be found by moving right 8 units and up 6 units from C, and by moving left 8 units and down 6 units from C.



Therefore, possible locations for X are $(6+8, 8+6) = (14, 14)$ and $(6-8, 8-6) = (-2, 2)$, which are the possible coordinates of the points S and T.

Method 2

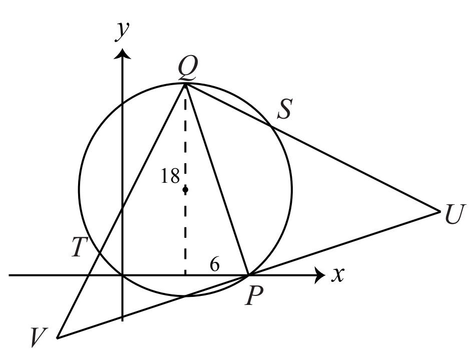
The slope of the line segment connecting $P(12, 0)$ and $C(6, 8)$ is $\frac{8-0}{6-12} = \frac{8}{-6} = -\frac{4}{3}$.
Since XC is perpendicular to PC, then the slope of XC is the negative reciprocal of $-\frac{4}{3}$, which equals $\frac{3}{4}$.
Therefore, the line through X and C has slope $\frac{3}{4}$ and passes through $C(6, 8)$, and so has equation $y - 8 = \frac{3}{4}(x-6)$ or $y = \frac{3}{4}x - \frac{18}{4} + 8$ or $y = \frac{3}{4}x + \frac{7}{2}$.
To find the possible locations of the point X on the circumference of the circle, we substitute $y = \frac{3}{4}x + \frac{7}{2}$ into the equation of the circle to obtain the following equivalent equations:

$$\begin{aligned}
(x-6)^2 + \left(\tfrac{3}{4}x + \tfrac{7}{2} - 8\right)^2 &= 100 \\
(x-6)^2 + \left(\tfrac{3}{4}x - \tfrac{9}{2}\right)^2 &= 100 \\
(x-6)^2 + \left(\tfrac{3}{4}(x-6)\right)^2 &= 100 \\
(x-6)^2 \left(1 + \left(\tfrac{3}{4}\right)^2\right) &= 100 \\
(x-6)^2 \left(\tfrac{25}{16}\right) &= 100 \\
(x-6)^2 &= 64 \\
x - 6 &= \pm 8
\end{aligned}$$

Therefore, $x = 8 + 6 = 14$ or $x = -8 + 6 = -2$, which give $y = \frac{42}{4} + \frac{7}{2} = 14$ and $y = \frac{-6}{4} + \frac{7}{2} = 2$, respectively.
The two possible locations for X are thus $(14, 14)$ and $(-2, 2)$, which are then the possible coordinates of the points S and T.

Solution 2

Consider the points $P(12, 0)$ and $Q(6, 18)$.
To move from Q to P, we travel 6 units right and 18 units down.
Starting from P, if we move 18 units right and 6 units up to point U with coordinates $(30, 6)$, then $PU = PQ$ and PU is perpendicular to PQ. (These facts follow from arguments similar to those in Solution 1, Method 1).
Similarly, starting from P, if we move 18 units left and 6 units down to point V with coordinates $(-6, -6)$, then $PV = PQ$ and PV is perpendicular to PQ.
Thus, $\triangle QPU$ and $\triangle QPV$ are right-angled and isosceles.
This means that $\angle PQU = \angle PQV = 45°$ (each is an angle in a right-angled, isosceles triangle), and so the points S and T where QU and QV respectively intersect the circle will give $\angle PQS = \angle PQT = 45°$.



The slope of the line segment joining $Q(6, 18)$ and $U(30, 6)$ is $\frac{18-6}{6-30}$ or $-\frac{1}{2}$.

Thus, the equation of the line through Q and U is $y-18=-\frac{1}{2}(x-6)$ or $y=-\frac{1}{2}x+21$.
Rearranging, we obtain $x=42-2y$.
Substituting into the equation of the circle $(x-6)^2+(y-8)^2=100$, we obtain successively

$$\begin{aligned}(36-2y)^2+(y-8)^2&=100\\4y^2-144y+1296+y^2-16y+64&=100\\5y^2-160y+1260&=0\\y^2-32y+252&=0\\(y-18)(y-14)&=0\end{aligned}$$

The solution $y=18$ gives $x=42-2y=6$, which are the coordinates of Q.
The solution $y=14$ gives $x=42-2y=14$, which are the coordinates of the point S.

The slope of the line segment joining $Q(6,18)$ and $V(-6,-6)$ is $\frac{18-(-6)}{6-(-6)}$ or 2.
Thus, the equation of the line through Q and V is $y-18=2(x-6)$ or $y=2x+6$.
Substituting into the equation of the circle $(x-6)^2+(y-8)^2=100$, we obtain successively

$$\begin{aligned}(x-6)^2+(2x-2)^2&=100\\x^2-12x+36+4x^2-8x+4&=100\\5x^2-20x-60&=0\\x^2-4x-12&=0\\(x-6)(x+2)&=0\end{aligned}$$

The solution $x=6$ gives $y=2x+6=18$, which are the coordinates of Q.
The solution $x=-2$ gives $y=2x+6=2$, which are the coordintes of the point T.

Therefore, the coordinates of S and T are $(14,14)$ and $(-2,2)$.

3. (a) By definition $a_2+b_2\sqrt{6}=(\sqrt{3}+\sqrt{2})^4$.
Now

$$(\sqrt{3}+\sqrt{2})^2=(\sqrt{3})^2+2(\sqrt{3})(\sqrt{2})+(\sqrt{2})^2=3+2\sqrt{6}+2=5+2\sqrt{6}$$

and so

$$(\sqrt{3}+\sqrt{2})^4=\left((\sqrt{3}+\sqrt{2})^2\right)^2=(5+2\sqrt{6})^2=5^2+2(5)(2\sqrt{6})+(2\sqrt{6})^2=25+20\sqrt{6}+24=49+20\sqrt{6}$$

Thus, $a_2+b_2\sqrt{6}=49+20\sqrt{6}$ and so $a_2=49$ and $b_2=20$.
(It is true that there are no other integers c,d for which $c+d\sqrt{6}=49+20\sqrt{6}$. This follows from the fact that $\sqrt{6}$ is irrational.)

(b) *Solution 1*

Let n be a positive integer.

Since $a_n+b_n\sqrt{6}=(\sqrt{3}+\sqrt{2})^{2n}$ and $a_n-b_n\sqrt{6}=(\sqrt{3}-\sqrt{2})^{2n}$, then adding these equations, we obtain

$$(a_n+b_n\sqrt{6})+(a_n-b_n\sqrt{6})=(\sqrt{3}+\sqrt{2})^{2n}+(\sqrt{3}-\sqrt{2})^{2n}$$

or

$$2a_n=(\sqrt{3}+\sqrt{2})^{2n}+(\sqrt{3}-\sqrt{2})^{2n}$$

Now $\sqrt{3}-\sqrt{2}\approx 0.32$.

Since $0<\sqrt{3}-\sqrt{2}<1$, then for every positive integer m, we have $0<(\sqrt{3}-\sqrt{2})^m<1$.

In particular, $0<(\sqrt{3}-\sqrt{2})^{2n}<1$, and so

$$(\sqrt{3}+\sqrt{2})^{2n}+0<(\sqrt{3}+\sqrt{2})^{2n}+(\sqrt{3}-\sqrt{2})^{2n}<(\sqrt{3}+\sqrt{2})^{2n}+1$$

Since $2a_n=(\sqrt{3}+\sqrt{2})^{2n}+(\sqrt{3}-\sqrt{2})^{2n}$, then

$$(\sqrt{3}+\sqrt{2})^{2n}<2a_n<(\sqrt{3}+\sqrt{2})^{2n}+1$$

This gives $(\sqrt{3}+\sqrt{2})^{2n}<2a_n$ and $2a_n-1<(\sqrt{3}+\sqrt{2})^{2n}$, or, equivalently,

$$2a_n-1<(\sqrt{3}+\sqrt{2})^{2n}<2a_n$$

as required.

Solution 2

Since $(\sqrt{3}+\sqrt{2})^{2n}=a_n+b_n\sqrt{6}$ and $(\sqrt{3}-\sqrt{2})^{2n}=a_n-b_n\sqrt{6}$, then

$$(a_n+b_n\sqrt{6})(a_n-b_n\sqrt{6})=(\sqrt{3}+\sqrt{2})^{2n}(\sqrt{3}-\sqrt{2})^{2n}=((\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}))^{2n}=(3-2)^{2n}=1$$

In other words, $(a_n)^2-6(b_n)^2=1$.

Since a_n and b_n are positive integers, then $a_n=\sqrt{(a_n)^2}=\sqrt{1+6(b_n)^2}>1$.

Since $a_n>1$, then the following inequalities must successively be true:

$$\begin{array}{rcccl}
2-2a_n & < & 0 & < & 1 \\
-2a_n+1 & < & -1 & < & 0 \\
(a_n)^2-2a_n+1 & < & (a_n)^2-1 & < & (a_n)^2 \\
(a_n-1)^2 & < & 6(b_n)^2 & < & (a_n)^2 \\
a_n-1 & < & \sqrt{6}b_n & < & a_n \\
2a_n-1 & < & a_n+b_n\sqrt{6} & < & 2a_n \\
2a_n-1 & < & (\sqrt{3}+\sqrt{2})^{2n} & < & 2a_n
\end{array}$$

as required.

(c) Since $(\sqrt{3}+\sqrt{2})^2 = a_1+b_1\sqrt{6}$, then $3+2\sqrt{6}+2 = a_1+b_1\sqrt{6}$ which gives $a_1+b_1\sqrt{6} = 5+2\sqrt{6}$ and so $a_1 = 5$ and $b_1 = 2$.
In (a), we determined that $a_2 = 49$ and $b_2 = 20$.
Using (b), we see that $2a_1 - 1 < (\sqrt{3}+\sqrt{2})^2 < 2a_1$ or $9 < (\sqrt{3}+\sqrt{2})^2 < 10$. This means that $(\sqrt{3}+\sqrt{2})^2$ is larger than 9 but smaller than 10, and so its units digits is 9. (A calculator can tell us that $(\sqrt{3}+\sqrt{2})^2 \approx 9.899$.)
Similarly, $2a_2 - 1 < (\sqrt{3}+\sqrt{2})^4 < 2a_2$ or $97 < (\sqrt{3}+\sqrt{2})^4 < 98$. This means that $(\sqrt{3}+\sqrt{2})^4$ is between 97 and 98, and so its units digit is 7.
In general, because $2a_n - 1$ and $2a_n$ are integers that satisfy $2a_n - 1 < (\sqrt{3}+\sqrt{2})^{2n} < 2a_n$, then d_n (the units digit of $(\sqrt{3}+\sqrt{2})^{2n}$) is the same as the units digit of $2a_n - 1$.
Now for every positive integer k,

$$(\sqrt{3}+\sqrt{2})^{2(k+1)} = (\sqrt{3}+\sqrt{2})^{2k}(\sqrt{3}+\sqrt{2})^2$$

and so

$$a_{k+1} + b_{k+1}\sqrt{6} = (a_k + b_k\sqrt{6})(5+2\sqrt{6}) = (5a_k + 12b_k) + (2a_k + 5b_k)\sqrt{6}$$

Since $a_{k+1}, b_{k+1}, a_k, b_k$ are all integers and $\sqrt{6}$ is irrational, then

$$a_{k+1} = 5a_k + 12b_k \qquad \text{and} \qquad b_{k+1} = 2a_k + 5b_k$$

Using these relationships, we can make a table of the first several values of $a_n, b_n, 2a_n-1, d_n$:

n	a_n	b_n	$2a_n - 1$	d_n
1	5	2	9	9
2	49	20	97	7
3	485	198	969	9
4	4801	1960	9601	1
5	47 525	19 402	95 049	9

(For example, we see that $a_3 = 5a_2 + 12b_2 = 5(49) + 12(20) = 245 + 240 = 485$ and $b_3 = 2a_2 + 5b_2 = 2(49) + 5(20) = 198$.)
For each n, we would like to determine the value of d_n, which is the units digit of $2a_n - 1$.
Now the units digit of $2a_n - 1$ is completely determined by the units digit of a_n. Also, for each $n \geq 2$, $a_n = 5a_{n-1} + 12b_{n-1}$, so the units digit of a_n is completely determined by the units digits of a_{n-1} and b_{n-1}. In other words, it is sufficient to track the units digits of a_n and b_n.
We reconstruct the previous table now, keeping only units digits:

n	Units digit of a_n	Units digit of b_n	Units digit of $2a_n - 1$	d_n
1	5	2	9	9
2	9	0	7	7
3	5	8	9	9
4	1	0	1	1
5	5	2	9	9

Since the units digits of a_5 and b_5 are the same as the units digits of a_1 and b_1, and because each row in the table is completely determined by the previous row, then the units digits of a_n and b_n will repeat in a cycle of length 4. (Since $a_1 = a_5$ and $b_1 = b_5$, and a_6 and b_6 are determined from a_5 and b_5 in exactly the same way as a_2 and b_2 were determined from a_1 and b_1, then we must have $a_6 = a_2$ and $b_6 = b_2$. This argument continues to all subsequent terms.)

Since the units digits of a_n repeat in a cycle of length 4, then d_n will repeat in a cycle of length 4.

Note that when we divide 1867 by 4, we obtain a quotient of 466 and a remainder of 3.

Therefore,

$$\begin{aligned} d_1 + d_2 + d_3 + \cdots + d_{1865} + d_{1866} + d_{1867} &= 466(d_1 + d_2 + d_3 + d_4) + (d_1 + d_2 + d_3) \\ &= 466(9 + 7 + 9 + 1) + (9 + 7 + 9) \\ &= 466(26) + 25 \\ &= 12\,141 \end{aligned}$$

as required.