2005 Hypatia Contest (Grade 11)
Wednesday, April 20, 2005

1. For numbers a and b, the notation $a \diamond b$ means $a^2 - 4b$. For example, $5 \diamond 3 = 5^2 - 4(3) = 13$.

   (a) Evaluate $2 \diamond 3$.

   (b) Find all values of k such that $k \diamond 2 = 2 \diamond k$.

   (c) The numbers x and y are such that $3 \diamond x = y$ and $2 \diamond y = 8x$.
   Determine the values of x and y.

2. Gwen and Chris are playing a game. They begin with a pile of toothpicks, and use the following rules:

   - The two players alternate turns
   - On any turn, the player can remove 1, 2, 3, 4, or 5 toothpicks from the pile
   - The same number of toothpicks cannot be removed on two different turns
   - The last person who is able to play wins, regardless of whether there are any toothpicks remaining in the pile

   For example, if the game begins with 8 toothpicks, the following moves could occur:

   Gwen removes 1 toothpick, leaving 7 in the pile
   Chris removes 4 toothpicks, leaving 3 in the pile
   Gwen removes 2 toothpicks, leaving 1 in the pile

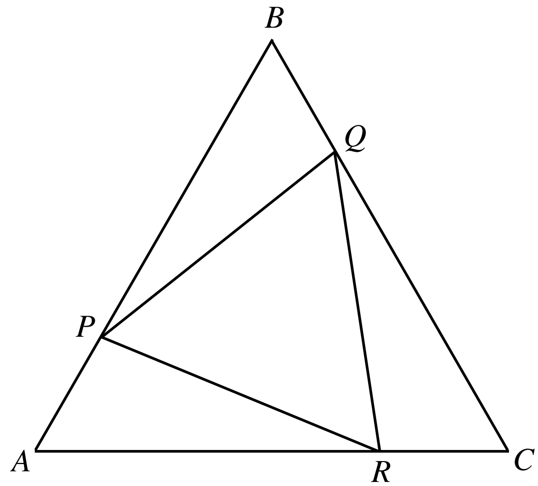
   Gwen is now the winner, since Chris cannot remove 1 toothpick. (Gwen already removed 1 toothpick on one of her turns, and the third rule says that 1 toothpick cannot be removed on another turn.)

   (a) Suppose the game begins with 11 toothpicks. Gwen begins by removing 3 toothpicks. Chris follows and removes 1. Then Gwen removes 4 toothpicks. Explain how Chris can win the game.

   (b) Suppose the game begins with 10 toothpicks. Gwen begins by removing 5 toothpicks. Explain why Gwen can always win, regardless of what Chris removes on his turn.

   (c) Suppose the game begins with 9 toothpicks. Gwen begins by removing 2 toothpicks. Explain how Gwen can always win, regardless of how Chris plays.

3. In the diagram, $\triangle ABC$ is equilateral with side length 4. Points P, Q and R are chosen on sides AB, BC and CA, respectively, such that $AP = BQ = CR = 1$.



(a) Determine the exact area of $\triangle ABC$. Explain how you got your answer.

(b) Determine the exact areas of $\triangle PBQ$ and $\triangle PQR$. Explain how you got your answers.

4. An *arrangement* of a set is an ordering of all of the numbers in the set, in which each number appears exactly once. For example, 312 and 231 are two of the possible arrangements of $\{1, 2, 3\}$.

(a) Determine the number of triples (a, b, c) where a, b and c are three different numbers chosen from $\{1, 2, 3, 4, 5\}$ with $a < b$ and $b > c$. Explain how you got your answer.

(b) How many arrangements of $\{1, 2, 3, 4, 5, 6\}$ contain the digits 254 consecutively in that order? Explain how you got your answer.

(c) A *local peak* in an arrangement occurs where there is a sequence of 3 numbers in the arrangement for which the middle number is greater than both of its neighbours.
For example, the arrangement 35241 of $\{1, 2, 3, 4, 5\}$ contains 2 local peaks.
Determine, with justification, the average number of local peaks in all 40 320 possible arrangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.