Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2004 Solutions
Hypatia Contest (Grade 11)

1. (a) *Solution 1*

Factoring the given equation,

$$x^2 + 5x + 6 = 0$$
$$(x+2)(x+3) = 0$$

so the roots are $x = -2$ and $x = -3$.

Solution 2

Using the quadratic formula,

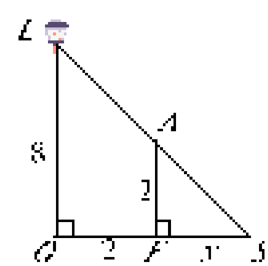
$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2(1)} = \frac{-5 \pm \sqrt{1}}{2}$$

so $x = \frac{-5+1}{2} = -2$ or $x = \frac{-5-1}{2} = -3$.

(b) When we increase the roots (–2 and –3) from (a) by 7, we get 5 and 4.
A quadratic equation that has 5 and 4 as roots is $(x-5)(x-4) = 0$ or $x^2 - 9x + 20 = 0$.

(c) First, we need to find the roots of this equation.
Since $x - 4$ is a factor, then $x = 4$ is a root.
So we must find the roots of $3x^2 - x - 2 = 0$. We can do this either by factoring the left side or by using the quadratic formula.
The easiest way is by factoring. We get $3x^2 - x - 2 = (3x+2)(x-1)$.
Therefore, the second and third roots are $x = -\frac{2}{3}$ and $x = 1$.
When we add 1 to each of the three roots, we obtain 5, $\frac{1}{3}$ and 2, so an equation having these three roots is $(x-5)(3x-1)(x-2) = 0$ or $(x-5)\left(3x^2 - 7x + 2\right) = 0$ or $3x^3 - 22x^2 + 37x - 10 = 0$. (Of course, there are many other equations that have these three numbers as roots.)

2. (a) In the diagram, *L* is the top of the lamp-post, *O* is the base of the lamp-post, *A* is the top of Alan's head, *F* is the point on the ground where Alan is standing, and *S* is the tip of Alan's shadow.



We know that *LO* and *AF* are perpendicular to *SO*, and *LAS* is a straight line.
Therefore, we see that triangle *LOS* is similar to triangle *AFS*, since they have a common angle and each has a right angle.

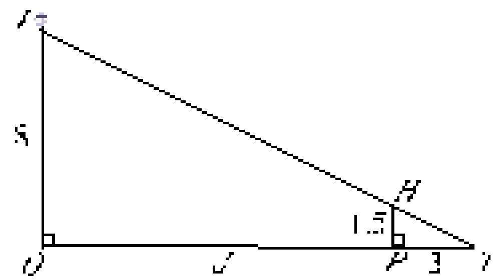
Therefore, $\dfrac{LO}{SO} = \dfrac{AF}{SF}$, or

$$\frac{8}{2+x} = \frac{2}{x}$$
$$8x = 4 + 2x$$
$$6x = 4$$
$$x = \tfrac{2}{3}$$

Therefore, Alan's shadow has length $\frac{2}{3}$ m.

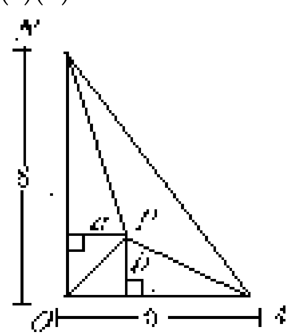
(b) In the new diagram, L and O are as before, H is the top of Bobbie's head, P is the point on the ground where Bobbie is standing, and T is the tip of Bobbie's shadow.
As in (a), we have that triangle LOT is similar to triangle HPT, and so, if d is the distance from the lamp to where Bobbie is standing (ie. the length of OP), then

$$\frac{LO}{TO} = \frac{HP}{TP}$$
$$\frac{8}{d+3} = \frac{1.5}{3}$$
$$1.5d + 4.5 = 24$$
$$1.5d = 19.5$$
$$d = 13$$



Therefore, Bobbie is standing 13 m from the lamp-post.

3. (a) Since triangle OMN is right-angled, its area is $\frac{1}{2}(OM)(ON) = \frac{1}{2}(8)(6) = 24$.
For the areas of triangles POM, PON and PMN all to be equal, they each must equal 8, ie. one-third of the total area.
Consider triangle POM. Its base has length 6 (the length of OM) and its height has length b (the distance from P to OM), so its area is $\frac{1}{2}(6)(b) = 3b$. For the area of triangle POM to be 8, we must have $b = \frac{8}{3}$.

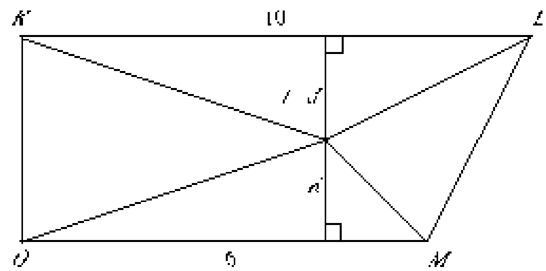


Consider now triangle PON. Its base has length 8 (the length of ON) and its height has length a (the distance from P to ON), so its area is $\frac{1}{2}(8)(a) = 4a$. For the area of triangle PON to be 8, we must have $a = 2$.
So if P has coordinates $\left(2, \frac{8}{3}\right)$, then each of triangles POM and PON has area 8, and so triangle PMN must also have area 8, since the area of the whole triangle OMN is 24.

(b) First, we calculate the area of quadrilateral *OMLK*.

OMLK is a trapezoid with *OM* parallel to *KL*, and so has area equal to the average of the bases times the height, ie. $\frac{1}{2}(10+6)t = 8t$.

So for the areas of the triangles *QOM*, *QML*, *QLK*, and *QKO* to be all equal, each of these four areas must be equal to $2t$, since the sum of these four areas is the area of the whole quadrilateral.



Consider triangle *QOM*. Its base has length 6 (the length of *OM*) and its height has length d (the distance from Q to *OM*), so its area is $\frac{1}{2}(6)(d) = 3d$. For the area of triangle *QOM* to be $2t$, we must have $d = \frac{2}{3}t$.

Consider next triangle *QLK*. Its base has length 10 (the length of *LK*) and its height has length $\frac{1}{3}t$ (the distance from Q to *LK*, since Q has y-coordinate $\frac{2}{3}t$). Thus, the area of triangle *QLK* is $\frac{1}{2}(10)\left(\frac{1}{3}t\right) = \frac{5}{3}t$, which is not equal to $2t$. (If we tried to set $\frac{5}{3}t = 2t$, we would then get $\frac{1}{3}t = 0$ or $t = 0$, which is not possible since we are told that $t > 0$.)

So it is impossible for the areas of both triangles *QOM* and *QLK* to be equal to $2t$.

Therefore, there is no point Q so that the areas of all four triangles area equal.

4. (a) We solve this problem by considering all of the possible cases. We will use the notation (G, Y, R) to denote the number of green (G), yellow (Y) and red (R) balls remaining. For example, the initial position is $(1,1,2)$.

If the green and yellow are chosen at the beginning, we then get $(0,0,3)$, so all of the remaining balls are red.

If the green and a red ball are chosen at the beginning, we then get $(0,2,1)$, so the next choice must be one yellow and one red, leaving $(1,1,0)$, and so the final two balls are chosen, leaving $(0,0,1)$, and so the final ball is red.

Similarly, the yellow and a green ball are chosen at the beginning, we then get $(2,0,1)$, so the next choice must be one green and one red, leaving $(1,1,0)$, and so the final two balls are chosen, leaving $(0,0,1)$, and so the final ball is red.

Thus, in all cases, the colour of the remaining ball or balls is always red.

(b) Here, we could again proceed by cases, but the number of cases would quickly get very large, so we should look for a better approach.

We notice that when a "move" is made (that is, two balls are removed and one is replaced), the parity of all three colours of balls changes. This is because the number of balls of each colour is being increased by 1 or decreased by 1, and so changes from odd to even or even to odd.
Therefore, since all three parities change together, the number of green balls and the number of red balls must always be both even or both odd, and the number of yellow balls is of the opposite parity. (Green and red start out both odd, after one move they will be both even, and so on; yellow starts out even, after one move it will be odd, and so on.)
When the process finally finishes, two of the numbers of balls will be 0, and so both be even. The only two colours which could have the same parity are green and red, so at the end there are 0 green and 0 red balls, so the colour of the remaining ball or balls is always yellow.
(Note that since the number of balls *decreases* on each turn, we are sure to actually end the game after at most 11 repetitions of the process.)

(c) *Solution 1*
In this version of the game, the total number of balls does not change, since the number of balls removed on each turn (2) is equal to the number of balls replaced on each turn (2).
Therefore, if the process were to end eventually with only one colour of ball remaining, then there would be 12 of that colour and 0 of each of the other two colours.
Consider $G-Y$ the difference between the number of balls that are green and the number of balls that are yellow. If the game got to 12 of one colour and 0 of the other two, then $G-Y$ would be equal to 12, 0 or –12.
We know that $G-Y$ is initially equal to –1. We will show that no matter what happens on each step of the process, $G-Y$ will always change by 0, 3 or –3 (ie. a multiple of 3). Once we have shown this, we can conclude that $G-Y$ can never be equal to 12, 0 or –12, since we cannot add and subtract 3's to –1 to get a multiple of 3. This will show that it is impossible for all of the remaining balls to be the same colour.
Let's suppose that at some point, the number of green balls is g and the number of yellow balls is y. After the next turn:

i) if a green and a yellow are replaced by two reds, there are $g-1$ green balls and $y-1$ yellow balls, so the difference is $(g-1)-(y-1)=g-y$, the same as its previous value
ii) if a green and a red are replaced by two yellows, there are $g-1$ green balls and $y+2$ yellow balls, so the difference is $(g-1)-(y+2)=g-y-3$
iii) if a yellow and a red are replaced by two greens, there are $g+2$ green balls and $y-1$ yellow balls, so the difference is $(g+2)-(y-1)=g-y+3$

So the difference always changes by 0, –3 or 3, so starting from –1, $G-Y$ can never become 0, so it is impossible for all of the balls to be the same colour.

Solution 2
Since all of the balls were eventually all of the same colour, then there would be 12 of one colour and 0 of each of the two remaining colours (since the number of balls does not change).

Suppose we were able to reach this state. In this solution, we will list the number of balls remaining by decreasing size (that is, we won't worry about which number goes with which colour). We will show by working backwards from the end, that it is impossible to get to 5,4,3.

When the process is reversed, we subtract 2 from one of the colours and add 1 to each of the others.
Thus, 12,0,0 can only come from 10,1,1.
10,1,1 can come from only 8,2,2.
8,2,2 can come from 6,3,3 or 9,3,0.
9,3,0 can come from 7,4,1 or 10,1,1.
7,4,1 can come from 5,5,2 or 8,2,2.
5,5,2 can come from 6,6,0 or 6,3,3.
6,6,0 can come from only 7,4,1.
6,3,3 can come from 4,4,4 or 7,4,1.
4,4,4 can come from only 5,5,2.
This creates a loop of possibilities: 12,0,0; 10,1,1; 9,3,0; 8,2,2; 7,4,1; 6,6,0; 6,3,3; 5,5,2; 4,4,4. It is impossible starting at 12,0,0 to get a position not in this list.

Thus, starting with 5,4,3 it is impossible to get to 12,0,0.