Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

1997 Solutions
Fermat Contest (Grade 11)

for the

NATIONAL BANK OF CANADA

Awards

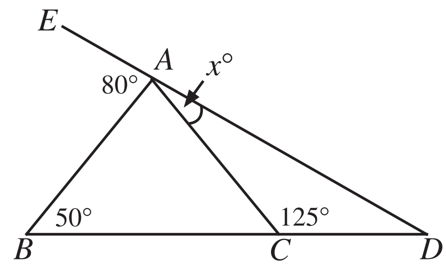
PART A:

1. ***Solution***

$$(1)^{10}+(-1)^8+(-1)^7+(1)^5 = 1+1-1+1$$
$$= 2$$

ANSWER: (E)

2. ***Solution***



Since B, C and D lie on a straight line,

$$\angle BCA = 180 - 125$$
$$= 55$$

In ΔABC, $\angle BAC = 180-(50+55)$

$$= 75$$

Since E, A and D lie on a straight line,

$$x = 180-(80+75)$$
$$= 25$$

The value of x is 25.

ANSWER: (C)

3. ***Solution***

If the first day is a Monday, then every seventh day is also a Monday, and Monday falls on the following numbered days: 1, 8, 15, 22, 29, 36, 43.

In 45 consecutive days, the maximum number of Mondays is seven.

ANSWER: (C)

4. ***Solution***

Let x represent the number.

Then $x\left(x^2\right)\left(\frac{1}{x}\right) = \frac{100}{81}$

$$x^2 = \frac{100}{81}$$
$$x = \pm\frac{10}{9}$$

Since x is positive, $x = \frac{10}{9}$.

ANSWER: (D)

5. ***Solution***

Since the sum of the seven integers is 77, their average is $\frac{77}{7} = 11$.

Because there is an odd number of consecutive integers, 11 is the middle integer and 14 is the largest.

ANSWER: (C)

6. ***Solution***

The product of $2E3$ and $3E2$ is $(2\times10^3)(3\times10^2)=6\times10^5$, which can be written as $6E5$.

ANSWER: (B)

7. ***Solution***

Draw in AB, perpendicular to CD at B.

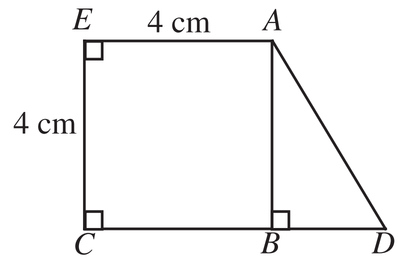
Since $ABCE$ is a square, each side has length 4 cm. Thus, $BD=3$ cm.

Using Pythagoras in ΔABD gives

$$AD^2=4^2+3^2$$

$$AD=5.$$

The perimeter of the figure is $4+4+5+7=20$ cm.

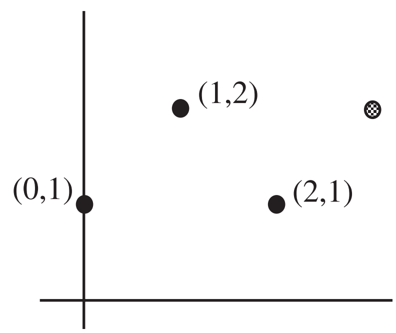


ANSWER: (E)

8. ***Solution***

There are three possible locations for the fourth vertex, but each resulting parallelogram has the same area.

Locating the fourth vertex at $(3,2)$ yields a parallelogram with base 2 and height 1. Its area is $(2)(1)=2$.



ANSWER: (B)

9. ***Solution***

The largest value of $\frac{x^2}{2y}$ will occur when the numerator is largest and the denominator is smallest.

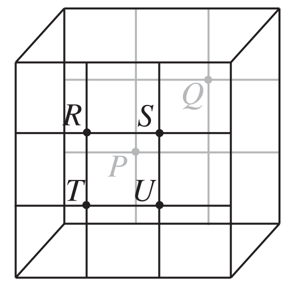
Since $10\le x\le 20$, the largest value of x^2 is $(20)^2=400$. Similarly, since $40\le y\le 60$, the smallest value of $2y$ is $2(40)=80$.

Thus, the largest value of $\frac{x^2}{2y}$ is $\frac{400}{80}=5$.

ANSWER: (A)

10. ***Solution***

When the cube is folded together, the points are located as shown in the diagram. The point S is diametrically opposite P.



ANSWER: (C)

PART B:

11. ***Solution***

Since the middle integer (the median) is 83, the two larger integers must both be 85 (the mode). Since the range of the integers is 70, the smallest integer is $85-70=15$.
The sum of all five integers is $5(69)=345$.
Thus, the second smallest integer is $345-(85+85+83+15)=77$. ANSWER: (A)

12. ***Solution***

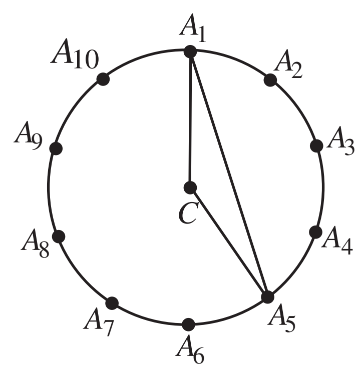
Join A_1A_5, A_1C, and A_5C, as shown.
Since the points $A_1, A_2, A_3, \ldots, A_{10}$ are equally spaced, they generate equal angles at C, each of measure $\frac{360}{10}=36$.

Thus, $\angle A_1CA_5 = 4(36)$
$= 144$

Since $A_1C = A_5C$ (radii), then ΔA_1CA_5 is isosceles and

$$\angle A_1A_5C = \frac{(180-144)}{2} = 18$$

Thus, the value of $\angle A_1A_5C$ is 18 degrees.



ANSWER: (A)

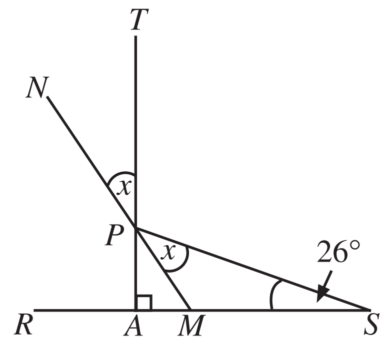
13. ***Solution***

The first twelve numbers in the list begin with either the digit 1 or 2. The next six begin with the digit 3. In order, these six numbers are 3124, 3142, 3214, 3241, 3412, 3421.
We see that the number 3142 is in the fourteenth position. ANSWER: (B)

14. ***Solution***

In the diagram, extend TP to meet RS at A. Since $AT \perp RS$, then $\angle SPA = 180° - 90° - 26°$

$$= 64°$$



Label points M and N. Since $\angle TPN$ and $\angle MPA$ are vertically opposite angles, they are equal in size, so $\angle MPA = x$.

But $\angle SPA = 2x$, so

$$2x = 64°$$

$$x = 32°.$$

Thus, the value of x is $32°$. ANSWER: (D)

15. ***Solution***

Multiplying the given equations together yields

$$(x^2yz^3)(xy^2) = (7^3)(7^9)$$

$$x^3y^3z^3 = 7^{12}$$

Taking the cube root of each side of the equation gives $xyz = 7^4$. ANSWER: (E)

16. ***Solution***

An even integer is found by taking the previous odd integer and adding 1.

Thus, the sum of the first 50 positive even numbers is obtained by finding the sum of the first 50 positive odd integers then adding 50 1's.

The sum of the first 50 positive even integers is $50^2 + 50$. ANSWER: (D)

17. ***Solution***

Let s represent the population of Sudbury at the beginning of 1996, and v represent the population of Victoria at the beginning of 1996.

At the end of 1996, Sudbury's population was $0.94s$ and Victoria's population was $1.14v$, where $0.94s = 1.14v \quad (1)$.

To find s:v, rearrange (1) to obtain

$$\frac{s}{v} = \frac{1.14}{0.94} = \frac{57}{47}$$

ANSWER: (B)

18. ***Solution***

First count the number of integers between 3 and 89 that can be written as the sum of exactly two elements. Since each element in the set is the sum of the two previous elements, 55 can be added to each of the seven smallest elements to form seven unique integers smaller than 89.

In the same way, 34 can be added to each of the seven smaller elements, 21 can be added to each of the six smaller elements, and so on.

The number of integers between 3 and 89 that can be written as the sum of two elements of the set is $7+7+6+5+4+3+2=34$. Since there are 85 integers between 3 and 89, then $85-34=51$ integers cannot be written as the sum of exactly two elements in the set.

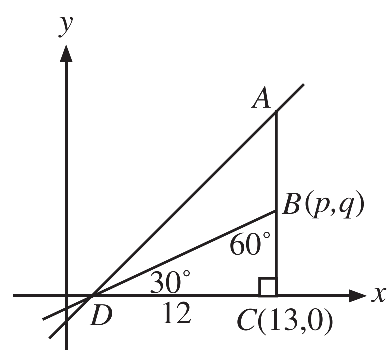
ANSWER: (E)

19. ***Solution***

Since D is the x-intercept of line AD, the coordinates of D are $(1,0)$. Thus, $DC=12$.

The slope of AD is $\sqrt{3}$, so $AC=12\sqrt{3}$ and $\angle ADC=60°$. Since BD bisects $\angle ADC$, then $\angle BDC=30°$ and ΔDBC is a $30°-60°-90°$ triangle.

Thus, $BC=\frac{12}{\sqrt{3}}$ and this is the value of q.



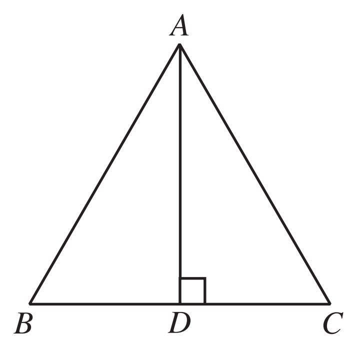
ANSWER: (D)

20. ***Solution***

In the diagram, there are 27 black triangles. If the entire diagram was divided into the smallest size equilateral triangles, there would be

$8+2(7)+2(6)+2(5)+2(4)+2(3)+2(2)+2(1)=64$

(counting by rows). Thus, $\frac{27}{64}$ of ΔABC is coloured black.



Drop a perpendicular from A, meeting BC at D. Since ΔABC is equilateral and $AB=16$, then $BD=DC=8$. Using either the Pythagorean Theorem or the fact that ΔABD is a $30°-60°-90°$ triangle, we find that $AD=8\sqrt{3}$.

Thus, the area of ΔABC is $\frac{1}{2}\left(8\sqrt{3}\right)(16)=64\sqrt{3}$, and the area of all the black triangles is $\frac{27}{64}\left(64\sqrt{3}\right)=27\sqrt{3}$.

ANSWER: (C)

PART C:

21. ***Solution***

Simplify the expression $\dfrac{\left(\frac{a}{c}+\frac{a}{b}+1\right)}{\left(\frac{b}{a}+\frac{b}{c}+1\right)}=11$ as follows:

$$\frac{\left(\dfrac{ab+ac+bc}{bc}\right)}{\left(\dfrac{bc+ab+ac}{ac}\right)}=11$$

$$\frac{ac}{bc}=11$$

$$\frac{a}{b}=11 \quad (\text{since } c \neq 0)$$

$$a=11b$$

By substitution, the condition $a+2b+c \le 40$ becomes $13b+c \le 40$.
Since b and c are positive integers, then b can only take on the values 1, 2, or 3. The values of a correspond directly to the values of b, since $a=11b$.
If $b=3$, there is one corresponding value of c. When $b=2$, there are 14 possible values of c. Finally if $b=1$, there are 27 possible values of c.
Therefore, the number of different ordered triples satisfying the given conditions is $1+14+27=42$.

ANSWER: (D)

22. ***Solution***

Rewrite the expression $2x^2-2xy+y^2=289$ as $x^2+(x-y)^2=289$. Using Pythagorean triples, the possible values for x are 0, 8, 15, and 17.
Substituting each value of x and solving for the corresponding values of y yields seven different ordered pairs.

ANSWER: (B)

23. ***Solution***

Using $f(x)=px+q$, calculate the following:

$$\begin{aligned} f(f(x)) &= f(px+q) \\ &= p(px+q)+q \\ &= p^2x+pq+q \end{aligned}$$

and

$$\begin{aligned} f(f(f(x))) &= f\left(p^2x + pq + q\right) \\ &= p\left(p^2x + pq + q\right) + q \\ &= p^3x + p^2q + pq + q \end{aligned}$$

Equate this to $8x + 21$ to find $p = 2$ and $q = 3$.
Thus, $p + q = 5$. ANSWER: (C)

24. ***Solution***
This is an example of a telescoping series.
Substitute $n = 2, 3, 4, ..., 50$ into the given expression to obtain

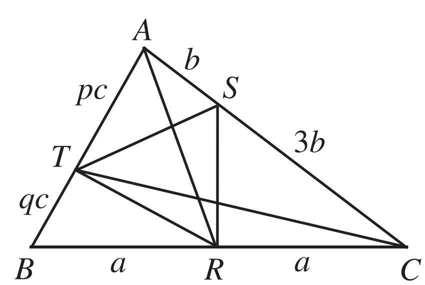
$$\begin{aligned} t_2 - t_1 &= 7 \\ t_3 - t_2 &= 9 \\ t_4 - t_3 &= 11 \\ &\vdots \\ t_{49} - t_{48} &= 101 \\ t_{50} - t_{49} &= 103 \end{aligned}$$

Adding the left and right sides separately gives

$$\begin{aligned} t_2 - t_1 + t_3 - t_2 + \cdots + t_{50} - t_{49} &= 7 + 9 + 11 + \cdots 101 + 103 \\ t_{50} - t_1 &= (7 + 103) + (9 + 101) + \cdots + (53 + 57) + 55 \\ &= 24(110) + 55 \\ &= 2695 \end{aligned}$$

Since $t_1 = 5$, then $t_{50} = 2700$. ANSWER: (A)

25. ***Solution***
Join A to R and C to T. Label the diagram as shown. Let the area of ΔABC be k.



Since w is the area of ΔCRS, and triangles CRS and ARC have equal heights and bases that are in the ratio $3b : 4b = 3 : 4$, then

$$w = \frac{3}{4} \text{ (area of } \Delta ARC)$$

However, triangles ARC and ABR also have equal heights and bases that are in the ratio $a : a = 1 : 1$, so

$$\begin{aligned} w &= \tfrac{3}{8} \text{ (area of } \Delta ABC) \\ &= \tfrac{3}{8}k \end{aligned}$$

Similarly, $x = \dfrac{q}{(p+q)}$ (area of ΔABR)

$$= \frac{q}{2(p+q)} \text{ (area of } \Delta ABC)$$

$$= \frac{q}{2(p+q)} k$$

and $z = \frac{1}{4}$ (area of ΔATC)

$$= \frac{p}{4(p+q)} \text{ (area of } \Delta ABC)$$

$$= \frac{p}{4(p+q)} k$$

Since $x^2 = wz$, then $\frac{q^2k^2}{4(p+q)^2} = \frac{3pk^2}{32(p+q)}$ which simplifies to $3p^2 + 3pq - 8q^2 = 0$.

Dividing through by q^2 yields a quadratic in $\frac{p}{q}$, the desired ratio.

Solving this quadratic gives $\frac{p}{q} = \frac{\sqrt{105} - 3}{6}$. ANSWER: (E)