Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

# 1997 Solutions
# Cayley Contest (Grade 10)

for the
NATIONAL BANK OF CANADA
Awards

© 1997 Waterloo Mathematics Foundation

## PART A:

1. ***Solution***

$$2\tfrac{1}{10} + 3\tfrac{11}{100} = 2.1 + 3.11$$
$$= 5.21$$

ANSWER: (D)

2. ***Solution***

$$(1)^{10} + (-1)^{8} + (-1)^{7} + (1)^{5} = 1 + 1 - 1 + 1$$
$$= 2$$

ANSWER: (C)

3. ***Solution***

Since the final result contains a factor of 10, it must have at least one zero at the end. The only listed possibility is 30. ANSWER: (E)

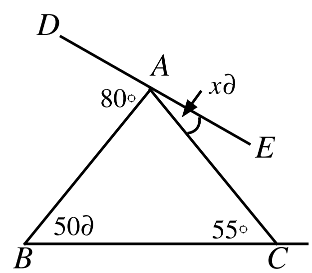
4. ***Solution***

If the first day is a Monday, then every seventh day is also a Monday, and Monday falls on the following numbered days: 1, 8, 15, 22, 29, 36, 43.
In 45 consecutive days, the maximum number of Mondays is seven. ANSWER: (C)

5. ***Solution***

The value of $-BAC$, in degrees, is $180 - 50 - 55 = 75$.
Since $D$, $A$, and $E$ lie on a straight line,

$$80 + 75 + x = 180$$
$$x = 25$$




ANSWER: (A)

6. ***Solution***

The first ten balloons are popped in the following order: *C*, *F*, *I*, *L*, *D*, *H*, *A*, *G*, *B*, and *K*.
The two remaining balloons are *E* and *J*. ANSWER: (D)

7. ***Solution***

In rectangle *ABCD*, side *AB* has length $4 - (-3) = 7$. Since the area of the rectangle is 70, the length of side *AD* must be $\frac{70}{7} = 10$.
Thus, the value of $k$ is $1 + 10 = 11$. ANSWER: (D)

8. ***Solution***

Rearranging and combining the inequalities yields $p < q < t < r < s$. The greatest of these numbers is $s$. ANSWER: (B)

9. ***Solution***

Since the sum of the seven integers is 77, their average is $\frac{77}{7} = 11$. Because there is an odd number of consecutive integers, 11 is the middle integer and the smallest is 8. ANSWER: (C)

10. ***Solution***

The greatest possible value of $p^q$ is $3^4 = 81$.

Thus, the greatest possible value of $p^q + r^s$ is $3^4 + 2^1 = 83$. ANSWER: (E)

## PART B:

11. ***Solution***

Since $x$, $y$, $z$, and $w$ are integers, then $y$ must divide evenly into both 6 and 25. The only possible value of $y$ is 1. Thus, $x = 6$ and $w = 25$. The value of $xw$ is $(6)(25) = 150$. ANSWER: (A)

12. ***Solution***

Let the depth of each cut be $d$.

Then,

$$\begin{aligned} 80(15) - 5d - 15d - 10d &= 990 \\ 1200 - 30d &= 990 \\ 30d &= 210 \\ d &= 7 \end{aligned}$$

The depth of each cut is 7. ANSWER: (B)

13. ***Solution***

Using Pythagoras in $\Delta ABC$ gives $BC$ to be 6. Since $BC = 3DC$, $DC = 2$.

$$\begin{aligned} AD^2 &= 2^2 + 8^2 \\ &= 68 \end{aligned}$$

Using Pythagoras again in $\Delta ADC$, $AD = \sqrt{68}$ ANSWER: (E)

14. ***Solution***

The first twelve numbers in the list begin with either the digit 1 or 2. The next six begin with the digit 3. In order, these six numbers are 3124, 3142, 3214, 3241, 3412, 3421.

We see that the number 3142 is in the fourteenth position. ANSWER: (B)

15. ***Solution***

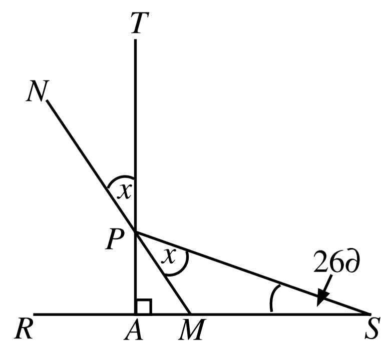
Since each factor of 10 produces a zero at the end of the integer we want to know how many 10's occur in the product.

The product of $20^{50}$ and $50^{20}$ can be rewritten as follows:

$$\begin{aligned}\left(20^{50}\right)\left(50^{20}\right) &= \left(2^2 \times 5\right)^{50}\left(5^2 \times 2\right)^{20}\\ &= 2^{100} \times 5^{50} \times 5^{40} \times 2^{20}\\ &= 2^{120} \times 5^{90}\\ &= 2^{30}\left(2^{90} \times 5^{90}\right)\\ &= 2^{30} \times 10^{90}\end{aligned}$$

From this, we see that there are 90 zeros at the end of the resulting integer. ANSWER: (C)

16. ***Solution***




In the diagram, extend $TP$ to meet $RS$ at $A$. Since $AT \perp RS$,

$$\angle SPA = 180° - 90° - 26°$$

then

$$= 64°$$

Label points $M$ and $N$. Since $\angle TPN$ and $\angle MPA$ are vertically opposite angles, they are equal, so $\angle MPA = x$.

Since

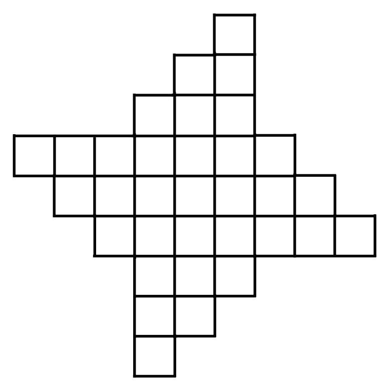
$$\angle SPA = 2x, \quad 2x = 64°$$

$$x = 32°$$

Thus, the value of $x$ is $32°$.

ANSWER: (A)

17. ***Solution***



Since all of the shorter edges are equal in length, the diagram can be subdivided into 33 small squares, as shown. Each of these squares has area $\frac{528}{33} = 16$ and the length of each side is $\sqrt{16} = 4$.

By counting, we find 36 sides and a perimeter of 144.

ANSWER: (E)

18. ***Solution***

Rewrite $\frac{30}{7}$ as

$$4+\frac{2}{7} = 4+\frac{1}{\left(\frac{7}{2}\right)}$$
$$= 4+\frac{1}{3+\frac{1}{2}}$$

By comparison, $x = 4$, $y = 3$ and $z = 2$.

Thus, $x + y + z = 9$. ANSWER: (B)

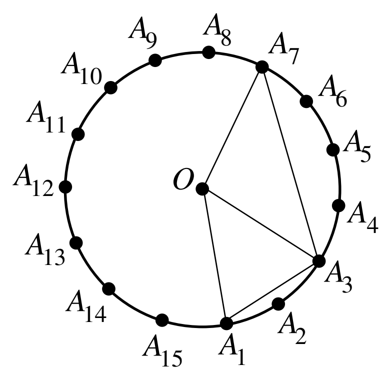
19. ***Solution***

By multiplying the given equations together we obtain

$$\left(x^2yz^3\right)\left(xy^2\right) = \left(7^4\right)\left(7^5\right)$$
$$x^3y^3z^3 = 7^9$$

Taking the cube root of each side gives $xyz = 7^3$. ANSWER: (C)

20. ***Solution***




Join $A_1, A_3$ and $A_7$ to $O$, the centre of the circle, as shown. Since the points $A_1, A_2, A_3, \ldots, A_{15}$ are evenly spaced, they generate equal angles at $O$, each of measure $\frac{360^\circ}{15} = 24^\circ$.

Thus, $\angle A_1OA_3 = 48^\circ$ and $\angle A_3OA_7 = 96^\circ$.

Since $OA_1 = OA_3$ (radii), $\Delta A_1OA_3$ is isosceles, and

$$\angle A_1A_3O = \frac{\left(180^\circ - 48^\circ\right)}{2}$$
$$= 66^\circ.$$

Similarly, $\Delta A_3OA_7$ is isosceles, and

$$\angle OA_3A_7 = \frac{\left(180^\circ - 96^\circ\right)}{2}$$
$$= 42^\circ.$$

$$\angle A_1A_3A_7 = \angle A_1A_3O + \angle OA_3A_7$$
$$= 66^\circ + 42^\circ$$

Thus, $= 108^\circ$. ANSWER: (D)

## PART C:

21. ***Solution***

Simplify the expression $\dfrac{\left(\frac{a}{c}+\frac{a}{b}+1\right)}{\left(\frac{b}{a}+\frac{b}{c}+1\right)} = 11$ as follows:

$$\frac{\left(\frac{ab+ac+bc}{bc}\right)}{\left(\frac{bc+ab+ac}{ac}\right)}=11$$

$$\frac{ac}{bc}=11$$

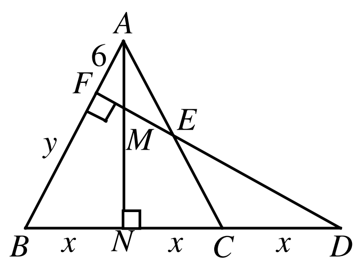
$$\frac{a}{b}=11 \quad \text{(since } c \neq 0\text{)}$$

$$a=11b$$

By substitution, the condition $a+2b+c \le 40$ becomes $13b+c \le 40$.
Since *b* and *c* are positive integers, then *b* can only take on the values 1, 2, or 3. The values of *a* correspond directly to the values of *b*, since $a=11b$.
If $b=3$, there is one corresponding value of *c*. When $b=2$, there are 14 possible values of *c*. Finally if $b=1$, there are 27 possible values of *c*.
Therefore, the number of different ordered triples satisfying the given conditions is $1+14+27=42$.

ANSWER: (D)

22. ***Solution***




Drop a perpendicular from $A$ to $BC$, and label as shown.
Since $\triangle ABC$ is equilateral, $BN=NC=CD$. Let $BN=x$ and $BF=y$.
Then $6+y=2x$. (1)
Also, $\angle FAM=30^\circ$, and $\triangle AMF$ is a $30^\circ-60^\circ-90^\circ$ triangle with sides in the ratio $1:\sqrt{3}:2$.
Thus, $AM=4\sqrt{3}$ and $FM=2\sqrt{3}$.
Use similar triangles *DBF* and *AMF* to find

$$\frac{DB}{BF}=\frac{AM}{MF}$$

$$\frac{3x}{y}=\frac{4\sqrt{3}}{2\sqrt{3}}$$

$$3x=2y \qquad (2)$$

Solving equations (1) and (2) yields $x=12$ and $y=18$.
Find $AN=12\sqrt{3}$; the area of $\triangle ABC$ is thus $144\sqrt{3}$.
Use similar triangles *EAF* and *AMF* to find $FE=6\sqrt{3}$; the area of $\triangle AFE$ is $18\sqrt{3}$.
The area of quadrilateral $FBCE$ is $126\sqrt{3}$.

ANSWER: (C)

23. ***Solution***

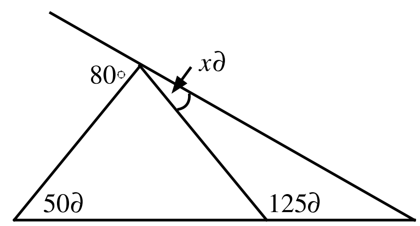
First count the number of integers between 3 and 89 that can be written as the sum of exactly two elements. Since each element in the set is the sum of the two previous

elements, 55 can be added to each of the seven smallest elements to form seven unique integers smaller than 89.
In the same way, 34 can be added to each of the seven smaller elements, 21 can be added to each of the six smaller elements, and so on.
The number of integers between 3 and 89 that can be written as the sum of two elements of the set is $7+7+6+5+4+3+2=34$. Since there are 85 integers between 3 and 89, then $85-34=51$ integers cannot be written as the sum of exactly two elements in the set.

ANSWER: (A)

24. ***Solution***
Since exactly five interior angles are obtuse, then exactly five exterior angles are acute and the remaining angles must be obtuse. Since we want the maximum number of obtuse angles, assume that the five acute exterior angles have a sum less than 90∂. Since obtuse angles are greater than 90∂, we can only have three of them.
Thus, the given polygon can have at most $3+5=8$ sides.

ANSWER: (B)




25. ***Solution***
Join $A$ to $R$ and $C$ to $T$. Label the diagram as shown. Let the area of $DABC$ be $k$.
Since triangles CRS and CRA have equal heights and bases that are in the ratio $3b:4b=3:4$, then
area of $DCRS=\frac{3}{4}$ (area of $DCRA$)
However, since triangles $CRA$ and $ABR$ also have equal heights and bases that are in the ratio $a:a=1:1$, then

$$DCRS=\tfrac{3}{8}\text{ (area of }DABC)$$

$$=\frac{3k}{8}$$

area of

Similarly, area of $DTBR=\dfrac{q}{p+q}$ (area of $DABR$)

$$=\frac{qk}{2(p+q)}$$

and area of $DATS=\frac{1}{4}$ (area of $DATC$)

$$=\frac{pk}{4(p+q)}$$

Since (area of $DRST$) = 2(area of $DTBR$), then

$$k-\frac{3k}{8}-\frac{qk}{2(p+q)}-\frac{pk}{4(p+q)}=\frac{2qk}{2(p+q)}$$

$$1-\frac{3}{8}-\frac{q}{2(p+q)}-\frac{p}{4(p+q)}=\frac{2q}{2(p+q)}$$ (since $k\,\Sigma 0$)

Simplify this expression to get $\frac{p}{q}=\frac{7}{3}$. ANSWER: (E)